International Journal of Theoretical Physics, Vol. 42, No. 6, June 2003 (© 2003)

Aspects of Scalar Brane-World Cosmological Perturbations

Cédric Deffayet[1,2]

Received April 14, 2003

We summarize an approach to deal with scalar brane-world cosmological perturbations based on Mukohyama's master equation. We also give its relation to one based on perturbing the effective Einstein's equations on the brane (involving the *Weyl fluid*).

KEY WORDS: brane mechanics; cosmological perturbations.

1. INTRODUCTION

The phenomenology of brane-world models has been the subject of intensive investigations in the last years. Its richness is in part due to the fact that these models can lead to modifications of gravity at small, but almost macroscopic (Antoniadis *et al.*, 1998; Arkani-Hamed *et al.*, 1998; 1999; Randall and Sundrum, 1999), or even very large (cosmological) distances (Dvali *et al.*, 2000; Gregory *et al.*, 2000). Because of those modifications of gravity, the cosmology of brane worlds can differ dramatically from standard cosmology (Binetruy *et al.*, 2000; Deffayet, 2001) and can potentially lead to various ways to test them [as well as new ways to address old problems, such as the vDVZ discontinuity (Deffayet *et al.*, 2002b). This is particularly true with the advent of precision cosmological measurement. Conversely, the brane-world models can also lead to new scenarios for the primordial universe or its recent cosmological evolution.[3] Measurement of the anisotropies of the Cosmic Microwave Background (CMB), and large-scale

[1] Department of Physics, Center for Cosmology and Particle Physics, New York University, New York, New York.

[2] To whom correspondence should be addressed at GReCO, Institut d'Astrophysique de Paris, 98 bis boulevard Arago, 75014 Paris, France; e-mail: deffayet@iap.fr.

[3] This is seen in the Dvali–Gabadadze–Porrati (DGP) model (Dvali *et al.*, 2000), which has the ability to produce acceleration of the universe, as suggested by SNIa data (Riess *et al.*, 1998), without the need for a nonzero cosmological constant (Deffayet, 2001; Deffayet *et al.*, 2002a) in a way currently in agreement with supernovae and Cosmic Microwave Background (CMB) data (Deffayet *et al.*, 2002c) (see also Avelino and Martins, 2002; Deffayet *et al.*, 2001).

galaxies or weak lensing surveys provide a unique way to test gravity at large scales (see, e.g., Binetruy and Silk, 2001; Uzan and Bernardeau, 2001) but also our comprehension of the physics of the primordial universe. As far as brane-world models are concerned, this relies in particular on a better understanding of scalar cosmological perturbations in these kinds of models. This subject has already been investigated by many authors (Boehm *et al.*, 2001; Bridgman *et al.*, 2001, 2002; Deruelle *et al.*, 2001; Deruelle and Dolezel, 2001; Deruelle and Katz, 2001; Dorca and van de Bruck, 2001; Garriga and Sasaki, 2000; Gordon and Maartens, 2001; Hawking *et al.*, 2000; Kodama *et al.*, 2000; Koyama and Soda, 2000; Langlois, 2000, 2001; Langlois *et al.*, 2001; Leong *et al.*, 2001; Maartens, 2000; Mukohyama, 2000a,b; 2001; Sago *et al.*, 2002; van de Bruck *et al.*, 2000a,b; van de Bruck and Dorca, 2000) with only limited results, as far as observable predictions are concerned (see, e.g., Bridgman *et al.*, 2002; Wands, 2002). We summarize here an approach that has the virtue of reducing the problem to solving a single hyperbolic equation in the bulk space–time [the master equation first derived by Mukohyama 2000b)] obeying a particular boundary condition on the brane that can be derived from the brane matter equation of state in the simplest cases (Deffayet, 2002; Kodama *et al.*, 2000). We also discuss the relation between this approach and one based on perturbing effective Einstein's equations on the brane [first obtained by Shiromizu *et al.* (2000)]. In the remaining of this introduction we introduce the latter equations and some features of the background cosmological solutions. We then turn to discuss cosmological perturbations.

1.1. Effective 4D Einstein's Equations in Brane World

We will consider in this paper a brane of codimension 1, namely a three-brane embedded in a 5 D bulk space–time. In such a case, it is particularly simple to obtain effective 4D Einstein's equations that can be then used to compare the homogeneous cosmology on the brane to a standard one, or to do the same comparison for cosmological perturbations. Let us first recall how these effective Einstein's equations can be obtained (Binetruy *et al.*, 2000; Shiromizu *et al.*, 2000). We first note that it is always possible to choose a, so-called Gaussian Normal (referred to as GN in the rest of this work), coordinate system where the 5D line element can be put in the form

$$ds^2 = dy^2 + g^{(4)}_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{1}$$

and the brane is the hypersurface defined by $y = 0$. In this coordinate system the 5D Einstein's equations are simply given by

$$G^{(5)}_{AB} = \kappa^2_{(5)}\delta(y)S_{\mu\nu}\delta^\mu_A\delta^\nu_B - \Lambda_{(5)}g^{(5)}_{AB}, \tag{2}$$

where $G^{(5)}_{AB}$ is the 5D Einstein's tensor, $\kappa^2_{(5)}$ is the inverse third power of the 5D reduced Planck mass, $\Lambda_{(5)}$ is the bulk cosmological constant, and $S_{\mu\nu}$ is the effective energy–momentum tensor for the brane. The latter is model-dependent, e.g., for the Randall–Sundrum model II (RS model in the following) (Randall and Sundrum, 1999), which is given by

$$S_{\mu\nu} = T^{(\mathrm{M})}_{\mu\nu} - \lambda_{(4)} g^{(4)}_{\mu\nu}, \tag{3}$$

where $T^{(\mathrm{M})}_{\mu\nu}$ is the *real* matter energy momentum tensor and $\lambda_{(4)}$ is the brane tension. In the brane-induced gravity model of Dvali–Gabadadze–Porrati (DGP model in the following) (Dvali *et al.*, 2000), where an Einstein–Hilbert term computed with the induced metric on the brane is present in the brane action, $S_{\mu\nu}$ is given by

$$S_{\mu\nu} = T^{(\mathrm{M})}_{\mu\nu} - \frac{1}{\kappa^2_{(4)}} G^{(4)}_{\mu\nu}, \tag{4}$$

where $\kappa^2_{(4)}$ is the inverse second power of the 4D reduced Planck mass. Einstein's equations (2) lead to Israel's junctions conditions (Darmois, 1927; Israel, 1966; Lanczos, 1924) that relate the jump of the extrinsic curvature tensor of the brane $K_{\mu\nu}$ to whatever distributional source appears on the right-hand side of (2) (here accounting for the brane-localized fields, *real matter*, or induced metric-dependent terms). Using a Gauss decomposition and the above Israel's conditions, one can then derive from Eq. (2) effective 4D Einstein's equations on the brane, relating the brane intrinsic curvature to its effective energy momentum content (Binetruy *et al.*, 2000). In particular one gets the following equation (Shiromizu *et al.*, 2000):

$$G^{(4)}_{\mu\nu} = -\frac{1}{2} g^{(4)}_{\mu\nu} \Lambda_{(5)} + \kappa^4_{(5)} \prod\nolimits_{\mu\nu} - \mathcal{E}_{\mu\nu}, \tag{5}$$

where $G^{(4)}_{\mu\nu}$ is the 4D Einstein tensor, $\prod_{\mu\nu}$ a tensor quadratic in the brane effective energy–momentum tensor $S_{\mu\nu}$, and $\mathcal{E}_{\mu\nu}$ is defined as the limiting value on the brane of the electric part of the bulk Weyl tensor. In the GN coordinate system (1) it is simply given by

$$\mathcal{E}_{\mu\nu} = C^5_{\mu 5 \nu}, \tag{6}$$

where C^A_{BCD} is the bulk Weyl tensor. One sees that $\mathcal{E}_{\mu\nu}$ acts in the effective Einstein's equations (5) as an external source with an energy momentum tensor $T^{(\mathcal{E})}_{\mu\nu}$ that one can define as $T^{(\mathcal{E})}_{\mu\nu} = -\mathcal{E}_{\mu\nu}/\kappa^2_{(4)}$. Following previous works (Bridgman *et al.*, 2002; Langlois, 2001; Langlois *et al.*, 2001; Martens, 2000), we will refer to this source as the *Weyl fluid* on the brane. One can verify that $T^{(\mathcal{E})}_{\mu\nu}$ is traceless, as follows from the tracelessness of the Weyl tensor, so that the Weyl fluid shares some similarities with a radiation fluid. From the so-called Codacci equation, one gets in addition conservation laws for the brane effective energy–momentum tensor. Namely, one has for the RS and DGP models

$$D^\mu S_{\mu\nu} = 0, \tag{7}$$

$$D^\mu T^{(\mathrm{M})}_{\mu\nu} = 0. \tag{8}$$

One also deduces from Eq. (5) and Bianchi identities that (Shiromizu *et al.*, 2000)

$$D^{\mu}\mathcal{E}_{\mu\nu} = \kappa_{(5)}^{4} D^{\mu} \prod\nolimits_{\mu\nu}, \tag{9}$$

which can be considered as a conservation equation for the Weyl fluid. In general the energy momentum tensor of the latter is indeed not conserved with respect to the induced metric, since the right-hand side of the above equation does not vanish; we will however still refer to the equations deduced from (9) as conservation equations for the Weyl fluid.

1.2. Background Cosmological Metric

Equations (5) can be used to derive the background (homogeneous) cosmology of the brane-world models under interest. It is indeed possible to obtain *exact* solutions for the metric in the bulk and on the brane describing an homogeneous cosmology (Binetrug *et al.*, 2000; Deffayet, 2001; Kraus, 1999). We will not need here the explicit form of those solutions, but only mention that the line elements of (background) space–times we will consider can be put in the form (in a GN coordinate system (1))

$$ds_{(5)}^2 = -n^2(t, y)\,dt^2 + a^2(t, y)\delta_{ij}\,dx^i\,dx^j + dy^2, \tag{10}$$

where the 3D metric δ_{ij} is a flat Euclidean metric (we will only consider here the case of a spatially flat universe). One can further choose a time parametrization such that the function n is set to 1 on the brane, in which case the induced metric is simply given by

$$ds_{(4)}^2 = -dt^2 + a_{(b)}^2\delta_{ij}\,dx^i\,dx^j$$

(where $a_{(b)} \equiv a(t, y = 0)$)[4] and is of FLRW form. Considering comoving observers to sit at fixed comoving coordinates x^i on the brane, t is then simply the cosmological time on the brane. Accordingly with the symmetries of (10) the effective energy momentum tensor of the brane is taken of the form

$$S_{\nu}^{\mu} = \delta(y)\,\mathrm{diag}(-\rho, P, P, P).$$

We point out that in the above expression ρ and P are not to be understood as *real matter* energy density and pressure, which will be denoted as $\rho_{(\mathrm{M})}$ and $P_{(\mathrm{M})}$ respectively, but as their *effective matter* counterparts. Their relation to real matter energy density and pressure are model-dependent, and can be obtained, for example from Eqs. (3) and (4). We will furthermore assume in the following that the background bulk is a slice of a maximally symmetric space–time (with vanishing Weyl tensor).

[4] Here and in the rest of the paper, the index (b) means that the value of the indexed quantity is taken on the brane.

2. SCALAR PERTURBATIONS AND THE MASTER EQUATION

Equations (5) can also be used to study the problem of scalar[5] cosmological perturbations in these models, simply by linearizing them around some known background. This task, followed by some authors, is however much more difficult than in the homogeneous case, namely because those equations do not close on the brane. This is because of the lack of a local evolution equation for all the Weyl degrees of freedom.[6] It is however possible to close a subsystem for large-scale cosmological perturbations (Maartens, 2000), but this does not enable to compute the Sachs–Wolfe effect (Langlois *et al.*, 2001). To obtain the evolution equations for cosmological perturbations on the brane, one should then solve the equations of motions for perturbations in the bulk. However, one should keep in mind the linearized equations (5), since they provide an easy and clear way to compare brane-world cosmological perturbations to standard ones. In the perspective of solving bulk equations of motions for perturbations, one may wish to use the work of Mukohyama who showed that the bulk scalar Einstein's equations are solved by a single master variable obeying a master equation (Mukohyama, 2000b), (this is reminded in subsection 2.1). The master variable can be related to brane matter by junction conditions (subsections 2.2 and 2.3) on the brane (Kodama *et al.*, 2000; Mukohyama, 2000a). We also discuss here some aspects of the relationship between Mukohyama's master equation and the perturbed effective Einstein's equations (5) on the brane, and in addition, how to obtain a boundary condition for the master variable on the brane (subsection 2.3).

2.1. Bulk Perturbations

As shown by Mukohyama (2000b) (see also Kodama *et al.*, 2000), the linearized scalar Einstein's equations in a maximally symmetric bulk can be conveniently solved introducing a master variable Ω obeying in the bulk (when Ω has a nontrivial dependence in the comoving coordinates x^i) the master equation that reads in a GN coordinate system (10)

$$\left(\frac{\dot{\Omega}}{na^3}\right)^{\cdot} + \left(\frac{\Lambda_{(5)}}{6} - \frac{\Delta}{a^2}\right)\frac{n\Omega}{a^3} - \left(\frac{n\Omega'}{a^3}\right)' = 0. \tag{11}$$

In this equation, Δ is defined by $\Delta = \delta^{ij}\partial_j\partial_i$, a prime means a derivative with respect to y, and a dot means a derivative with respect to t. In the rest of this paper, we will implicitly consider all the perturbations as Fourier transformed with respect

[5] The term scalar refers here to the scalar–vector–tensor decomposition, familiar to the standard (4D) theory of cosmological perturbations, with respect to the isometries of the 3D spatial sections parallel to the brane-world volume. The case of scalar perturbations (w.r.t. vector or tensor) is the most difficult case, but also the most interesting as far as phenomenology is concerned.

[6] Namely, for the Weyl fluid anisotropic stress.

to the x^i, in order to do a mode by mode analysis. In particular (11) can be rewritten as $\mathcal{D}_\Delta \Omega = 0$, where $\mathcal{D}_\Delta$ is a second-order hyperbolic differential operator acting on y- and t-dependent functions (in the GN system), and Δ is understood to be replaced by $-\vec{k}^2$, where $\vec{k}$ is the comoving momentum. Equation (11) is then only valid when $\vec{k}^2$ does not vanish (Mukohyama, 2000b), which is the only case of interest as far as cosmological perturbations are concerned.

The remarkable fact about Eq. (11) is that its solution enables to solve *all* the linearized scalar Einstein's equation in the bulk, which contain *four* gauge invariant scalar perturbations $\tilde{A}$, $\tilde{A}_y$, $\tilde{A}_{yy}$, and $\tilde{R}$. One can indeed take the most general scalar linearized perturbation around the background metric (10) of the form[7]

$$g_{AB} = \begin{pmatrix} -n^2(1+2\bar{A}) & a^2\bar{B}_{|i} & n\bar{A}_y \\ a^2\bar{B}_{|i} & a^2[(1+2\bar{\mathcal{R}})\delta_{ij} + 2\bar{E}_{|ij}] & a^2\bar{B}_{y|i} \\ n\bar{A}_y & a^2\bar{B}_{y|i} & 1+2\bar{A}_{yy} \end{pmatrix}, \tag{12}$$

where $_{|i}$ denotes a differentiation with respect to the comoving coordinate x^i. Out of those seven scalar perturbations, three (5D) scalar gauge transformations leave four gauge invariant perturbations (invariant with respect to 5D gauge transformations). The latter can be defined by (Bridgman *et al.*, 2002)

$$\begin{aligned}
\tilde{A} &= \bar{A} - \frac{1}{n}\left(\frac{a^2\bar{\sigma}}{n}\right)^{\cdot} + \frac{n'}{n}a^2\bar{\sigma}_y, \\
\tilde{A}_y &= \bar{A}_y + \frac{(a^2\bar{\sigma}_y)^{\cdot}}{n} + \frac{(a^2\bar{\sigma}')}{n} - 2\frac{n'}{n^2}a^2\bar{\sigma}, \\
\tilde{A}_{yy} &= \bar{A}_{yy} + (a^2\bar{\sigma}_y)', \\
\tilde{\mathcal{R}} &= \bar{\mathcal{R}} + aa'\bar{\sigma}_y - \frac{a\dot{a}}{n^2}\bar{\sigma},
\end{aligned}$$

where $\bar{\sigma} \equiv -\bar{B} + \dot{\bar{E}}$ and $\bar{\sigma}_y \equiv -\bar{B}_y + \bar{E}'$. Those gauge invariant metric perturbations are then related to Ω by (Bridgman *et al.*, 2002; Mukohyama, 2000)

$$\tilde{A} = -\frac{1}{6a}\left(2\Omega'' + \frac{1}{n^2}\ddot{\Omega} + \frac{\Lambda_{(5)}}{6}\Omega - \frac{\dot{n}}{n^3}\dot{\Omega} - \frac{n'}{n}\Omega'\right), \tag{13}$$

$$\tilde{A}_y = \frac{1}{an}\left(\dot{\Omega}' - \frac{n'}{n}\dot{\Omega}\right), \tag{14}$$

$$\tilde{A}_{yy} = \frac{1}{6a}\left(\Omega'' + \frac{2}{n^2}\ddot{\Omega} - \frac{\Lambda_{(5)}}{6}\Omega - 2\frac{\dot{n}}{n^3}\dot{\Omega} - 2\frac{n'}{n}\Omega'\right), \tag{15}$$

[7] We have put a bar on each linear scalar perturbation, in order to distinguish quantities in arbitrary gauge (barred expression), from those in the GNL (Gaussian Normal Longitudinal) gauge (the same expression with no bar), which we will use in most of this paper (see section 2.2.1).

$$\tilde{\mathcal{R}} = \frac{1}{6a}\left(\Omega'' - \frac{1}{n^2}\ddot{\Omega} + \frac{\Lambda_{(5)}}{6}\Omega + \frac{\dot{n}}{n^3}\dot{\Omega} + 2\frac{n'}{n}\Omega'\right). \tag{16}$$

In these equations, one clearly sees that one can consider Ω as a *potential* for the gauge invariant perturbations. On the other hand, one can express Ω as a function of the gauge invariant variables (Deffayet, 2002) by

$$\begin{aligned}\frac{2\dot{a}a'}{3a^3}\Delta^2\Omega = {} & \Lambda_{(5)}a^2\dot{a}(-2\dot{a}_{(b)}\tilde{A}_y + a'\tilde{A}_{yy} - a\tilde{A}') \\ & + \dot{a}a'\left(4\Delta(\tilde{A} + \tilde{A}_{yy}) + 6\dot{a}_{(b)}^2(\tilde{A} + 2\tilde{A}_{yy}) - 3\dot{a}_{(b)}a'\tilde{A}_y\right) \\ & + 6a\dot{a}_{(b)}^2(2\ddot{a}_{(b)}\tilde{A}_y + \dot{a}\tilde{A}' + 2a'\tilde{A}^{\cdot}) \\ & + 3a\dot{a}_{(b)}a'(2\ddot{a}_{(b)}(\tilde{A} - \tilde{A}_{yy}) + 2\dot{a}\tilde{A}_y' + a'\tilde{A}_y^{\cdot}),\end{aligned} \tag{17}$$

which shows in particular explicitly that Ω is 5D-gauge invariant.

2.2. Perturbations on the Brane

2.2.1. Induced Metric and Matter Perturbations

To deal with perturbations on the brane a convenient gauge choice can be made, namely one can simultaneously choose (Langlois, 2001)

$$A_y = 0, \tag{18}$$

$$B_y = 0, \tag{19}$$

$$A_{yy} = 0, \tag{20}$$

$$\xi = 0, \tag{21}$$

$$\sigma_{(b)} = 0, \tag{22}$$

where ξ refers to the perturbed y coordinate of the brane and the last equality defines the 4D longitudinal gauge on the brane. In the rest of this paper, we will always work in the above-defined gauge when dealing with quantities of which we take limiting values on the brane. We will call this gauge the Gaussian Normal Longitudinal gauge (GNL) and we have dropped bars to differentiate quantities in the GNL gauge from quantities in arbitrary gauge (e.g. A vs. $\bar{A}$). In the GNL gauge, the linearized 5D metric (12) is of GN form (1), while the brane sits in $y = 0$ and the induced metric on the brane is in the (4D) longitudinal form

$$ds^2 = -(1 + 2\Phi_{(b)})\,dt^2 + a^2(t)(1 - 2\Psi_{(b)})\delta_{ij}\,dx^i\,dx^j, \tag{23}$$

with $\Phi_{(b)}$ and $\Psi_{(b)}$ given by

$$\Phi_{(b)} = A_{(b)}, \tag{24}$$

$$\Psi_{(b)} = -\mathcal{R}_{(b)}. \tag{25}$$

Following the standard decomposition, one can define the scalar perturbations of the brane effective energy–momentum tensor by

$$\delta S_0^0 = -\delta\rho, \tag{26}$$

$$\delta S_i^0 = \delta q_{|i}, \tag{27}$$

$$\delta S_j^i = \delta P \delta_j^i + \left(\Delta_j^i - \frac{1}{3}\delta_j^i \Delta\right)\delta\pi, \tag{28}$$

where Δ_j^i is defined by

$$\Delta_j^i = \delta^{ik}\partial_k\partial_j,$$

so that one has $\Delta = \Delta_i^i$. Similar decomposition hold for the perturbations of the *real* matter and Weyl fluid energy momentum tensors, respectively $T_{\mu\nu}^{(\mathrm{M})}$ and $T_{\mu\nu}^{(\mathcal{E})}$.

2.2.2. *Junction Conditions*

The perturbations of effective matter on the brane are related to those of the metric through the linearization of Israel's junctions conditions. Those lead to the following expressions (Bridgman *et al.*, 2002)

$$A'_{(b)} = \frac{\kappa_{(5)}^2}{6}(3\delta P + 2\delta\rho), \tag{29}$$

$$\mathcal{R}'_{(b)} = \frac{1}{6}\kappa_{(5)}^2(\Delta\delta\pi - \delta\rho), \tag{30}$$

$$B'_{(b)} = \kappa_{(5)}^2 \frac{n_{(b)}^2}{a_{(b)}^2}\delta q, \tag{31}$$

$$E'_{(b)} = -\frac{1}{2}\kappa_{(5)}^2\delta\pi, \tag{32}$$

Similarly, one can get from Eq. (6) expressions relating the second y-derivatives on the brane of metric perturbations to Weyl fluids (and effective matter) perturbations. They read in the GNL gauge (Bridgman *et al.*, 2002)

$$\kappa_{(4)}^2\left(\delta P_{(\mathcal{E})} + \frac{2}{3}\delta\rho_{(\mathcal{E})}\right) = -\left\{A'' + 2\frac{n'}{n}A'\right\}_{(b)}, \tag{33}$$

$$\kappa_{(4)}^2\delta q_{(\mathcal{E})} = -\frac{1}{2}\frac{a_{(b)}^2}{n_{(b)}^2}\left\{B'' + \left(3\frac{a'}{a} - \frac{n'}{n}\right)B'\right\}_{(b)}, \tag{34}$$

$$\kappa_{(4)}^2 \delta\pi_{(\mathcal{E})} = \left\{ E'' + 2\frac{a'}{a} E' \right\}_{(b)}, \tag{35}$$

$$\kappa_{(4)}^2 \delta\rho_{(\mathcal{E})} = \left\{ 3\left(R'' + 2\frac{a'}{a} R' \right) + \Delta E'' + 2\frac{a'}{a} \Delta E' \right\}_{(b)}. \tag{36}$$

2.2.3. Perturbed Effective Einstein's Equations and the Master Equation

As we mentioned previously, it is desirable to compare the perturbed 4D effective Einstein's equations (5), reading, with obvious notations

$$\delta G_{\nu}^{(4)\mu} = -\delta \mathcal{E}_{\nu}^{\mu} + \kappa_{(5)}^4 \delta \prod\nolimits_{\nu}^{\mu}, \tag{37}$$

to the master equation. In particular the above equation involves the perturbations of the effective matter energy–momentum tensor $\delta\rho$, δq, δP, and $\delta\pi$ and the one of the Weyl fluid energy–momentum tensor $\delta\rho_{(\mathcal{E})}$, $\delta q_{(\mathcal{E})}$, and $\delta\pi_{(\mathcal{E})}$.[8] The relation between those degrees of freedom and the master variable is quite intricated, as we will see more clearly below. However, one can indeed recover (Deffayet, 2002) all the perturbed effective Einstein's equations (37) from the master equation (11) and the junction conditions for effective matter and Weyl fluid given above. This should not be a surprise since Eq. (5) has been obtained using the bulk equations of motion, and the latter are solved by the master variable obeying the master equation; but the backward derivation of the perturbed effective Einstein's equations (37) from the master equation (11) and the junction conditions for effective matter and Weyl fluid is quite tedious because of the derivative relation [exemplified by Eqs. (13)–(16)] between the master variable and the metric perturbations.

The outline of the derivation (see Deffayet, 2002, for more details) is to built "constrainst" valid everywhere in the bulk on the gauge invariant variables $\tilde{A}$, $\tilde{A}_y$, $\tilde{A}_{yy}$, and $\tilde{\mathcal{R}}$ (as well as on their y- and t-derivatives) out of the master equations and the definitions (13)–(16). Those last five equations can indeed be used to build overconstrained systems in Ω and its derivatives. The obtained constratints can be thought of as reconstruction of linearized 5D Einstein's equations (or linear combinations of the latter) in the bulk out of the master equation, in confirmation of the work by Mukohyama (2000b). One then takes limiting values on the brane of those constraints (i.e. values in $y \to 0$), where one replaces the gauge invariant variables by their expressions as functions of effective matter, Weyl fluid, and induced metric that can be obtained through the junction conditions. Those yield the sought-for components of equations (37). This means in particular that one does not need to "solve" equations (37) but rather only the bulk equation of motion to get the time evolution of perturbations on the brane. To do so, once suitable initial data are provided in the bulk, one needs a boundary condition on the brane for the master equation, the obtaining of which we now discuss.

[8] One has $\delta P_{(\mathcal{E})} = \delta\rho_{(\mathcal{E})}/3$.

2.3. Boundary Condition on the Brane

A first way to proceed would be to start from Eq. (17), and then use the junction conditions to get on the brane

$$\Delta^2\Omega_{(b)} = \left\{ -18a^4\dot{a}\left(\dot{\Psi} + \frac{\dot{a}}{a}\Phi\right) + 6a^3\Delta\Psi - \frac{\kappa_{(5)}^4}{2}a^5\rho\delta\rho \right\}_{(b)}. \quad (38)$$

A similar equation, as well as one for Ω', was obtained by Mukohyama (2001) and used by him to get an integro-differential equation for the effective matter and induced metric perturbation on the brane. This however is of no use to set a boundary condition for the master equation on the brane since the right-hand side of Eq. (38) is not known as a function of time. To obtain the boundary condition one needs an extra information on the matter localized on the brane, e.g., in the simplest case, the matter equation of state (Deffayet, 2002; Kodama *et al.*, 2000) as we now explain. Using Eqs. (11) and (13)–(16), as well as the junction conditions, one can get the following relations between the induced metric perturbation, effective matter perturbations, and the master variable (Deffayet, 2002):

$$\Phi_{(b)} = \frac{1}{6a_{(b)}}\left\{\left(\frac{2\Delta}{a^2} - \frac{\Lambda_{(5)}}{2}\right)\Omega + 6\frac{\dot{a}}{a}\dot{\Omega} - 3\ddot{\Omega} + \frac{\kappa_{(5)}^2}{2}(3P+4\rho)\Omega' \right.$$
$$\left. + \frac{\kappa_{(5)}^4}{2}a^3(3P+2\rho)\delta\pi\right\}_{(b)}, \quad (39)$$

$$\Psi_{(b)} = \frac{1}{6a_{(b)}}\left\{\left(\frac{\Delta}{a^2} - \frac{\Lambda_{(5)}}{2}\right)\Omega + 3\frac{\dot{a}}{a}\dot{\Omega} + \frac{\kappa_{(5)}^2}{2}\rho\Omega' + \frac{\kappa_{(5)}^4}{2}a^3\rho\delta\pi\right\}_{(b)}, \quad (40)$$

$$\delta\rho = \frac{1}{6a_{(b)}}\left\{\rho\frac{\Delta\Omega}{a^2} + 3\frac{\dot{a}}{a}(3P+2\rho)\dot{\Omega} + 6\frac{\Delta\Omega'}{\kappa_{(5)}^2 a^2} \right.$$
$$\left. - 18\frac{\dot{a}\Omega^{\cdot\prime}}{\kappa_{(5)}^2 a} + 6a\Delta\delta\pi - 18\dot{a}a\,(a\delta\pi)^{\cdot}\right\}_{(b)}, \quad (41)$$

$$\delta q = \frac{1}{6a_{(b)}}\left\{(3P+2\rho)\dot{\Omega} - \frac{6}{\kappa_{(5)}^2}\Omega^{\cdot\prime} - 6a^2(a\delta\pi)^{\cdot}\right\}_{(b)}, \quad (42)$$

$$\delta P = \frac{1}{6a_{(b)}}\left\{(P+\rho)\left(-\frac{2\Delta}{a^2} + \frac{\Lambda_{(5)}}{2}\right)\Omega - \left(3\dot{P} + \frac{\dot{a}}{a}(4\rho+6P)\right)\dot{\Omega} \right.$$
$$+ \rho\ddot{\Omega} - \frac{\kappa_{(5)}^2}{2}(P+\rho)(3P+4\rho)\Omega' + \frac{12}{\kappa_{(5)}^2}\frac{\dot{a}}{a}\Omega^{\cdot\prime} + \frac{6}{\kappa_{(5)}^2}\Omega^{\cdot\cdot\prime}$$
$$\left. - a^3\left[4\frac{\Delta}{a^2} + 4\Lambda_{(5)} + \frac{\kappa_{(5)}^4}{6}(9P^2 + 9P\rho + 4\rho^2)\right]\delta\pi + 6(a^3\delta\pi)^{\cdot\cdot}\right\}_{(b)}. \quad (43)$$

Similar expressions can be obtained for the perturbations of the Weyl fluid perturbations (Deffayet, 2002):

$$\delta\pi_{(\mathcal{E})} = \frac{1}{6\kappa_{(4)}^2 a_{(b)}^3}\left\{3\ddot{\Omega} - 3\frac{\dot{a}}{a}\dot{\Omega} - \frac{\Delta}{a^2}\Omega - \frac{3}{2}\kappa_{(5)}^2(P+\rho)\Omega'\right\}_{(b)}, \tag{44}$$

$$\delta\rho_{(\mathcal{E})} = \left\{\frac{\Delta^2\Omega}{3\kappa_{(4)}^2 a^5}\right\}_{(b)}, \tag{45}$$

$$\delta q_{(\mathcal{E})} = \left\{\frac{1}{3\kappa_{(4)}^2 a^3}\left(\frac{\dot{a}}{a}\Delta\Omega - \Delta\dot{\Omega}\right)\right\}_{(b)}. \tag{46}$$

One of the nice features of these equations is that one can verify inserting them expressions in the linearized components of equations (5), that the latter are identically satisfied whatever Ω. We wish here to stress that this means that the latter components do not contain more information than the above equations. Equations (39)–(46) can also be used to compute all the perturbations on the brane once Ω, and say $\delta\pi$, are known. Solving for Ω requires obtaining a boundary condition on the brane for the master variable. Let us now discuss how to do so in the simplest case of adiabatic perturbations of a perfect fluid. In this case, one has[9]

$$\delta\pi_{(\mathrm{M})} = 0. \tag{47}$$

This enables to express all the induced metric and real matter perturbations as functions of Ω and its derivatives through Eqs. (39)–(43). If one then considers adiabatic perturbations, which obey an equation of state of the form

$$\delta P_{(\mathrm{M})} = c_s^2 \delta\rho_{(\mathrm{M})}, \tag{48}$$

where c_s^2 is the sound velocity defined by

$$c_s^2 = \frac{\dot{P}_{(\mathrm{M})}}{\dot{\rho}_{(\mathrm{M})}}, \tag{49}$$

one gets from the replacement in Eq. (48) of $\delta\rho_{(\mathrm{M})}$ and $\delta P_{(\mathrm{M})}$ as functions of the master variable Ω a boundary condition for Ω. The latter takes the form (Kodama *et al.*, 2000)

$$F_{\mathrm{RS}}(\Omega)_{(b)} + G_{\mathrm{RS}}(\Omega')_{(b)} = 0, \tag{50}$$

where F_{RS} and G_{RS} are polynomials of the cosmic time-derivative ∂_t, with cosmic time-dependent coefficients that are known from Eqs. (41), (43), and (49),

[9] For the sake of simplicity, we will only discuss in the following the case of the RS model for which one has $\delta\rho_{(\mathrm{M})} = \delta\rho$, $\delta P_{(\mathrm{M})} = \delta P$, $\delta q_{(\mathrm{M})} = \delta q$, $\delta\pi_{(\mathrm{M})} = \delta\pi$. However, a similar discussion holds for the DGP model (Deffayet, 2002).

as well as from the background solution. A similar method can be used to obtain the boundary condition when the brane contains only a scalar field (Deffayet, in preparation) (the latter also obeys the perfect fluid condition (47); however, Eq. (48) is no longer valid). A boundary condition, of similar, albeit more complicated form, can be obtained in the same way for the model of Dvali–Gabadadze–Porrati, for perturbations of a perfect fluid with adiabatic (Deffayet, 2002) or scalar field perturbations (Deffayet, in preparation). The boundary condition (50) has an unconventional form, since it involves derivatives of Ω and Ω' along the brane. However, one can see that it leads to a well-posed problem once initial data are sufficiently well specified (Deffayet, in preparation). This has also be verified numerically in some simple cases (A. Lue, private communication).

3. CONCLUSIONS

Equations (11) and (50) should be then all what is needed to solve for the evolution of RS (respectively DGP) brane-world cosmological perturbations once initial conditions are supplied in the bulk. The issue of specifying initial data in the bulk, aside from the mathematical point of view which is discussed by Deffayet (2002; in preparation), is on the physical side a difficult task and requires some model for the primordial universe that can be applied to brane worlds. One sees then that Eqs. (11) and (50) play for the brane-world cosmological perturbations an equivalent role to the one played by the evolution equation for the gravitational potential (an ordinary differential equation for Φ, see, e.g., Bardeen, 1980; Kodama and Sasaki, 1984; Mukhanov *et al.*, 1992) for 4D adiabatic cosmological perturbations of a perfect fluid. Once Ω is known in the vicinity of the brane, one can compute all the induced metric, real matter, and Weyl fluid perturbations from Eqs. (39)–(46). We stress again here that one can verify that the real matter, induced metric, and Weyl fluid perturbations defined from those equations in terms of the master variable verify together identically (i.e. whatever the function Ω) the perturbed effective Einstein's equations (5) and conservation equations (8) and (9).

ACKNOWLEDGMENTS

I thank the organizers of the "Peyresq Physics 7" meeting on "Classical, Stochastic and Quantum gravity, String and Brane Cosmology" for having invited me, and having provided such a nice, instructive, and rich meeting. I also thank N. Deruelle, R. Durrer, G. Dvali, D. Langlois, A. Lue, K. Malik, M. Porrati, R. Scoccimarro, J. Shata, and M. Zaldarriaga for useful discussions. My work, while I was at NYU, was sponsored by NSF Award PHY 9803174 and by David and Lucile Packard Foundation Fellowship 99–1462.

REFERENCES

Antoniadis, I., Arkani-Hamed, N., Dimopoulos, S., and Dvali, G. R. (1998). *Physics Letters B* **436**, 257 (arXiv:hep-ph/9804398).

Arkani-Hamed, N., Dimopoulos, S., and Dvali, G. R. (1998). *Physics Letters B* **429**, 263 [arXiv:hep-ph/9803315].

Arkani-Hamed, N., Dimopoulos, S., and Dvali, G. R. (1999). *Physical Review D* **59**, 086004 (arXiv:hep-ph/9807344).

Avelino, P. P. and Martins, C. J. (2002). *Astrophysical Jorunal* **565**, 661 [arXiv:astro-ph/0106274].

Bardeen, J. M. (1980). *Physical Review D* **22**, 1882.

Binetruy, P., Deffayet, C., Ellwanger, U., and Langlois, D. (2000). *Physics Letters B* **477**, 285 [arXiv:hep-th/9910219].

Binetruy, P., Deffayet, C., and Langlois, D. (2000). *Nuclear Physics B* **565**, 269 [arXiv:hep-th/9905012].

Binetruy, P. and Silk, J. (2001). *Physical Review Letters* **87**, 031102 [arXiv:astro-ph/0007452].

Boehm, T., Durrer, R., and van de Bruck, C. (2001). *Physical Review D* **64**, 063504 [arXiv:hep-th/0102144].

Bridgman, H. A., Malik, K. A., and Wands, D. (2001). *Physical Review D* **63**, 084012 [hep-th/0010133].

Bridgman, H. A., Malik, K. A., and Wands, D. (2002). *Physical Review D* **65**, 043502 [arXiv:astro-ph/0107245].

Darmois, G. (1927). Mémorial des sciences mathématiques XXV.

Deffayet, C. (manuscript in preparation).

Deffayet, C. (2001). *Physics Letters B* **502**, 199 [hep-th/0010186].

Deffayet, C. (2002). *Physical Review D* **66**, 103504 [arXiv:hep-th/0205084].

Deffayet, C., Dvali. G., and Gabadadze, G. (2001). *Preprint* astro-ph/0106449.

Deffayet, C., Dvali, G. R., and Gabadadze, G. (2002a). *Physical Review D* **65**, 044023 [arXiv:astro-ph/0105068].

Deffayet, C., Dvali, G. R., Gabadadze, G., and Vainshtein, A. I. (2002b). *Physical Review D* **65**, 044026 [arXiv:hep-th/0106001].

Deffayet, C., Landau, S. J., Raux, J., Zaldarriaga, M., and Astier, P. (2002c). *Preprint* arXiv:astro-ph/0201164.

Deruelle, N. and Dolezel, T. (2001). *Physical Review D* **64**, 103506 [arXiv:gr-qc/0105118].

Deruelle, N., Dolezel, T., and Katz, J. (2001). *Physical Review D* **63**, 083513 [hep-th/0010215].

Deruelle, N. and Katz, J. (2001). *Physical Review D* **64**, 083515 [arXiv:gr-qc/0104007].

Dorca, M. and van de Bruck, C. (2001). *Nuclear Physics B* **605**, 215 [arXiv:hep-th/0012116].

Dvali, G., Gabadadze, G., and and Porrati, M. (2000). *Physics Letters B* **485**, 208 [arXiv:hep-th/0005016].

Garriga, J. and Sasaki, M. (2000). *Physical Review D* **62**, 043523 [hep-th/9912118].

Gordon, C. and Maartens, R. (2001). *Physical Review D* **63**, 044022 [hep-th/0009010].

Gregory, R., Rubakov, V. A., and Sibiryakov, S. M. (2000). *Physics Review Letters* **84**, 5928 [arXiv:hep-th/0002072].

Hawking, S. W., Hertog, T., and Reall, H. S. (2000). *Physical Review D* **62**, 043501 [hep-th/0003052].

Israel, W. (1966). *Nuovo Cimento B* **44S10**, 1 [Erratum in Nuovo Cimento. B **48**, (1967 NUCIA,B44,1.1966) 463].

Kodama, H., Ishibashi, A., and Seto, O. (2000). *Physical Review D* **62**, 064022 [arXiv:hep-th/0004160].

Kodama, H. and Sasaki, M. (1984). *Progress of Theoretical Physics Supplement* **78**, 1.

Koyama, K. and Soda, J. (2000). *Physical Review D* **62**, 123502 [hep-th/0005239].

Kraus, P. (1999). *JHEP* **9912**, 011 [arXiv:hep-th/9910149].

Lanczos, K. (1924). *Annals of Physics* **74**, 518; Sen, N. (1924). *Annals of Physics* **73**, 365.

Langlois, D. (2000). *Physical Review D* **62**, 126012 [hep-th/0005025].

Langlois, D. (2001). *Physical Review Letters* **86**, 2212 [arXiv:hep-th/0010063].

Langlois, D., Maartens, R., Sasaki, M., and Wands, D. (2001). *Physical Review D* **63**, 084009 [hep-th/0012044].
Leong, B., Dunsby, P., Challinor, A., and Lasenby, A. (2001). *Preprint* arXiv:gr-qc/0111033.
Maartens, R. (2000). *Physical Review D* **62**, 084023 [hep-th/0004166].
Maartens, R. (2001). *Preprint* arXiv:gr-qc/0101059.
Mukhanov, V. F., Feldman, H. A., and Brandenberger, R. H. (1992). *Physics Report* **215**, 203.
Mukohyama, S. (2001). *Physical Review D* **64** 064006 [arXiv:hep-th/0104185].
Mukohyama, S. (2000a). *Classical Quantum Gravity* **17**, 4777 [hep-th/0006146].
Mukohyama, S. (2000b). *Physical Review D* **62**, 084015 [arXiv:hep-th/0004067].
Perlmutter, S. *et al.*, (1999). (Supernova Cosmology Project Collaboration). *Astrophysical Journal* **517**, 565 [arXiv:astro-ph/9812133].
Randall, L. and Sundrum, R. (1999). *Physical Review Letters* **83**, 4690 [arXiv:hep-th/9906064].
Riess, A. G. *et al.*, (1998). *Astrophysical Journal* **116**, 1009.
Sago, N., Himemoto, Y., and Sasaki, M. (2002). *Physical Review D* **65**, 024014 [arXiv:gr-qc/0104033].
Shiromizu, T., Maeda, K. I., and Sasaki, M. (2000). *Physical Review D* **62**, 024012 [arXiv:gr-qc/9910076].
Uzan, J. P. and Bernardeau, F. (2001). *Physical Review D* **64**, 083004 [arXiv:hep-ph/0012011].
van de Bruck, C. and Dorca, M. (2000). *Preprint* hep-th/0012073.
van de Bruck, C., Dorca, M., Brandenberger, R. H., and Lukas, A. (2000a). *Physical Review D* **62**, 123515 [hep-th/0005032].
van de Bruck, C., Dorca, M., Martins, C. J., and Parry, M. (2000b). *Physics Letters B* **495**, 183 [hep-th/0009056].
Wands, D. (2002). *Preprint* arXiv:hep-th/0203107.